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The Stability of Primordial Magnetic Fields Produced by Phase Transitions

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Abstract

Primordial magnetic fields seem to be a generic relic of phase transitions in the early universe. We consider a primordial electromagnetic field formed as a result of a second-order phase transition, and show that it is stable to thermal fluctuations in the period immediately following. We also show how such a field arises in first order phase transitions. In both cases there is a transitive electric field produced during the transition.

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Introduction

Magnetic fields permeate most astrophysical systems, but their origin is unknown. The presence of such large-scale magnetic fields can affect processes like the formation of galaxies[1] and nucleosynthesis[2]. The existence of a galactic magnetic field, of order 10^{-6}G on a 100kpc scale, has been known for some time, but its origin is disputed. One theory generates it by means of amplifying a 'primordial' seed field as small as 10^{-20}G[1] via a galactic dynamo mechanism[4], but this leaves the question of where the seed field came from? There are a number of scenarios: by quantum fluctuations in an inflationary phase[5], by fluctuations inherent in the plasma[6] or as a result of a cosmological phase transition[7], such as that thought to decouple the electromagnetic and weak forces[8][9], or occur during inflation[10]. Here we consider the primordial magnetic field formed during the electro-weak phase transition.

To facilitate the analysis it is useful to employ the definition for the electromagnetic field strength introduced by 't Hooft[11] when studying magnetic monopoles.[2] This collapses to the standard definition if the Higgs field is taken to condense with the same phase throughout space. The important idea is that if the Higgs field possesses different phases in neighbouring regions of space then the resulting gradients in phase between these regions may produce a magnetic field. One needs to be careful, however, in case these fields are just a gauge artefact and can be removed via a suitable transformation.

Following the symmetry breaking produced by a phase transition the phase of the Higgs field is expected to vary randomly from horizon volume to horizon volume, through arguments of causality. In practice, space may be uncorrelated on a shorter scale, corresponding to the thermal correlation length, given by the inverse of the vector boson mass, m_V. Vachaspati[8] has pointed out that this will give rise to phase gradients between

[1]Recently Plaga[3] has devised a method to observe and measure magnetic fields as weak as 10^{-24}G in the inter-galactic voids, and so it may soon be possible to measure the primordial field directly.

[2]The original motivation for this was as a way of giving monopoles a magnetic field.

these correlation volumes, which, using 't Hooft's definition of the field strength, may in turn give rise to a magnetic field. What is more, for causally disconnected regions it is impossible to gauge these phase differences away. Thus, it is possible that a cosmological phase transition may well produce a physical magnetic field. What is an open question, however, is how persistent this field will be? Thermal fluctuations immediately following the phase transition and, later on, magnetohydrodynamical effects may both adversely affect the strength of the field. Here we shall consider only the effect of the fluctuations on the field strength since if this is too detrimental then there will be nothing left for the magnetohydrodynamics to work on. Such thermal fluctuations have not been taken into account in previous analyses.

In the first section we review 't Hooft's field strength definition in the context of the Weinberg-Salam model, and discuss previous attempts to estimate the root-mean-square field by means of statistical analyses. Previous work[8][9] only considered a second order phase transition. Here we also discuss the relative strength of the field produced by an equivalent first order transition. In the second part we analyse how stable such a field is to thermal fluctuations in the period immediately following a phase transition, the final section containing our conclusions and suggestions for future work.

Field Strength

In the Weinberg-Salam model, in which SU(2)$\times$U(1)$_Y$ gets broken down to U(1)$_{em}$ by the usual Higgs scalar, ϕ, the electromagnetic field strength can be written in the form[11]

$$\begin{aligned} F^{em}_{\mu\nu} &\equiv \sin(\theta_W) n^a F^a_{\mu\nu} + \cos(\theta_W) F^Y_{\mu\nu} \\ &\quad -i4g^{-1}\eta^{-2}\sin(\theta_W)[(D_\mu\phi)^\dagger D_\nu\phi - (D_\nu\phi)^\dagger D_\mu\phi] \end{aligned}$$

where $D_\mu = \partial_\mu - ieA^{em}_\mu$, $A^{em}_\mu = \sin(\theta_W) n^a W^a_\mu + \cos(\theta_W) B_\mu$ and $n^a = -2\phi^\dagger \sigma^a \phi/\eta^2$. A straightforward simplification of the right-hand-side yields

$$F^{em}_{\mu\nu} = \partial_\mu A^{em}_\nu - \partial_\nu A^{em}_\mu - i4g^{-1}\eta^{-2}\sin(\theta_W)[(\partial_\mu\phi)^\dagger\partial_\nu\phi - (\partial_\nu\phi)^\dagger\partial_\mu\phi] \tag{1}$$

which, it is seen, collapses to the usual form of the field strength if $n^a = (0,0,1)$ (corresponding to $\phi = (0,\eta)$).

As mentioned earlier, Vachaspati[8] suggested that gradients in the phase between causally disconnected regions can produce an electromagnetic field. Note that we can perform a gauge transformation on (1) to set the gauge fields to zero leaving all the information contained in the Higgs terms. Thus it is seen that it is possible to have a non-zero magnetic field even if the gauge fields are zero.

Setting $\phi = \exp(iT)\rho$ and $\rho = |\rho|\underline{l}$, where T is an element of the Lie algebra of SU(2)×U(1) and $\underline{l}$ is a constant, unit, complex two-vector,

$$\begin{aligned} F^{em}_{\mu\nu} = & -i4g^{-1}\eta^{-2}\sin\theta_W \Big[\rho^\dagger\left((\partial_\mu e^{-iT})(\partial_\nu e^{iT}) - (\partial_\nu e^{-iT})(\partial_\mu e^{iT})\right)\rho \\ & +2\rho^\dagger(\partial_\mu e^{-iT})e^{iT}\partial_\nu\rho - 2\rho^\dagger(\partial_\nu e^{-iT})e^{iT}\partial_\mu\rho\Big]. \end{aligned} \tag{2}$$

Since we expect the modulus of the scalar field to be uniform across space, to leading order, during a second order phase transition, the last two terms in the bracket will give non-zero contributions to F_{0i} only and hence not contribute to the magnetic field. They will produce an electric field however. This field is shortlived as it is driven by the rising $|\rho|$ and will drop to zero as $|\rho| \to \eta$, the global minimum of the effective potential, as the transition ends. A question worth considering is whether the magnetic field produced by the associated current (flowing across the boundary between uncorrelated domains) has sufficient time to freeze in on relevant scales. We mention several points concerning this electric field during our analysis, but concern ourselves, for the most part, with a study of the magnetic component of the field tensor, and leave a detailed study of the effect of the

transitive electric field aside for future work. In any case, the magnetic field induced by the transitive electric field is unlikely to be larger than the one considered here.

By taking the phase to perform a random walk of fixed step length along a line passing through N correlation volumes, Vachaspati estimated the gradient along the line to be $\partial_\mu \phi \sim \eta/(\sqrt{N}\xi)$ where η is the scale of breaking and $\xi \simeq m_V^{-1}$ is the correlation length at formation. On large scales the intergalactic plasma has a very large conductivity and the flux is frozen in to the co-moving vacuum. Using this we can estimate the size of the field at time t as $B_N \sim gT^2/4N$ where g is the SU(2) gauge coupling. On a 100kpc scale this gives a field today of magnitude $\sim 10^{-30}$G, much too small to give rise to the observed galactic field. Other statistical analyses are possible however. By assuming that the magnetic fields in neighbouring domains are uncorrelated Enqvist and Olesen[9] have managed to obtain a root-mean-square field strength of $B_{rms} \propto N^{-\frac{1}{2}}$ giving a primordial field today of $\sim 10^{-18}$G which is much more promising. Since this is a line average, and the galactic magnetic field is observed by Faraday rotation, this may well be a more realistic estimate. However, the question of whether it is legitimate to regard the flux in neighbouring cells as uncorrelated is still unanswered. For example, if we consider estimates calculated in a $N \times N \times N$ volume by taking the naïve model of the phase gradients performing a random walk through the N^3 cells, then we obtain an estimate proportional to $N^{-\frac{3}{2}}$ which is far too small. Clearly considerable work is still required before we can be sure that the estimates we are making, based on this mechanism for producing the field, are accurate.

Another feature of these estimates is that they assume a spinodal decomposition, and so, for the period we shall consider (and afterwards), there are no large regions of false vacuum. The magnetic field lies entirely within the true vacuum, along the boundaries between uncorrelated domains. In the case of a first order transition things are not so clear cut. Here the transition proceeds via tunnelling, bubbles of true vacuum nucleating in the false one, then expanding rapidly, colliding and coalescing with other bubbles of true vacuum

until they fill the Universe and the transition is complete.

Each time two bubbles collide, a ring of magnetic field is generated around the region of intersection. If the bubble radius on collision is of order ξ, then ξ is usually[13] greater than the thermal correlation length[3], and thus the phase gradient within two collided bubbles will in general be less than that within two neighbouring, uncorrelated domains in the equivalent second order phase transition. Because of this we would expect the magnetic field frozen in on large scales to be weaker than that produced by a second order transition. It is possible to quantify this slightly better.

As a crude estimate $\xi \simeq (v/\gamma)^{\frac{1}{4}}$ where v is the speed of the bubble wall and γ is the bubble nucleation rate per unit time per unit volume. For the electro-weak transition $v \simeq 0.1 - 1$ [13] whilst γ is related to the height of the barrier between the true and false vacua and hence the strength of the transition. Roughly speaking, the "stronger" the transition, the higher the barrier and, consequently, the lower the rate.

Stability of the Field

Having estimated the strength of the field that may be produced by a cosmological phase transition, we now turn to the question of how stable this field is to thermal fluctuations in the period immediately following its creation. To do this we adapt a method developed[12] to study the formation of topological defects. We restrict ourselves to the case of a second order transition. The field produced by a first order phase transition is likely to be more stable to thermal fluctuations.

From the electroweak Langrangian we can derive the equation of motion for the Higgs field,

$$(\partial_\mu + ieA^{em}_\mu)(\partial^\mu - ieA^{em\mu})\phi + \frac{\partial V}{\partial|\phi|^2}\phi \quad = \quad 0. \tag{3}$$

In [12] it was shown that the effect of gauge fields was small, and in fact only helped the

[3] There is a need for some caution here as for a slow transition some bubbles may nucleate at a lower temperature than others but for most cases this will be negligible.

stability of any fields present. Also, as mentioned above, we can make a transformation to move all the magnetic field information into the Higgs gradients. For these reasons it is justifiable to ignore the gauge fields.

As in the previous section, we make the substitution $\phi = \exp{(iT)}\rho$ where here, without loss of generality, we take $\rho = |\rho|(0, 1)$. This yields the following equation for ρ and T:

$$\partial_\mu \partial^\mu \rho + 2e^{-iT}(\partial_\mu e^{iT})\partial^\mu \rho + e^{-iT}(\partial_\mu \partial^\mu e^{iT})\rho + \frac{\partial V}{\partial \rho^2}\rho \quad = \quad 0. \tag{4}$$

To make further progress it is useful to make a couple of assumptions. Taking $T = \alpha \underline{m}.\underline{\sigma} + \beta$ for SU(2)×U(1) the first assumption is that α is constant and small, whilst the second is that we can ignore the U(1) symmetry (identical to assuming constant β). Thus we are constraining the degree to which the phase of the Higgs can vary. Our justification for doing this is partly practical and partly motivated by the fact that if we are to let any one part of the phase vary it has to be $\underline{m}$. If this is constant, then the resulting magnetic field is always zero. This is not true in the case of constant α and β - β in fact only affects the electric field. We have simply taken the minimal requirements to obtain a non-zero magnetic field.

Writing $\underline{\alpha} = \alpha \underline{m}$ we obtain

$$\begin{aligned} \partial_\mu \partial^\mu \rho - (\partial_\mu \underline{\alpha}.\partial^\mu \underline{\alpha})\rho + \frac{\partial V}{\partial \rho^2}\rho \quad &= \quad 0 \\ \partial_\mu \partial^\mu \underline{\alpha} + 2\partial_\mu \underline{\alpha} \partial^\mu \rho / \rho \quad &= \quad 0 \end{aligned} \tag{5}$$

where ρ now denotes $|\rho|$. As expected, the equations for α_1, α_2 and α_3 are identical - there is no preferred breaking direction. These equations are very similar to those derived in[12] and so we make heavy use of their perturbation analysis. To avoid repitition, full details can be obtained there, and here we sketch only the necessary detail for our analysis.

Consider the perturbed system where

$$\rho = \rho_B + \rho_F \quad , \quad \underline{\alpha} = \underline{\alpha}_B + \underline{\alpha}_F, \tag{6}$$

and anything with a subscript F is considered small. We have the following equations of motion:

$$\begin{aligned}\partial^2(\rho_B+\rho_F)-[\partial_\mu(\underline{\alpha}_B+\underline{\alpha}_F).\partial^\mu(\underline{\alpha}_B+\underline{\alpha}_F)](\rho_B+\rho_F)&\\ +(\rho_B+\rho_F)\left[\frac{\partial V}{\partial\rho^2}+\rho_F\frac{\partial}{\partial\rho}\left(\frac{\partial V}{\partial\rho^2}\right)\right] &= 0,\\ \partial^2(\underline{\alpha}_B+\underline{\alpha}_F)+2\partial_\mu(\underline{\alpha}_B+\underline{\alpha}_F)\partial^\mu(\rho_B+\rho_F)/(\rho_B+\rho_F) &= 0.\end{aligned}$$

Taking the zeroth-order in the perturbation expansion we obtain the background equation

$$\partial^2\rho_B-(\partial_\mu\underline{\alpha}_B.\partial^\mu\underline{\alpha}_B)\rho_B+\frac{\partial V(\rho)}{\partial\rho^2}\rho_B = 0. \tag{7}$$

Making use of the finite temperature form of the potential for a second order phase transition,

$$V(\phi,T) = \frac{1}{4}\lambda|\phi|^4-\frac{1}{2}(\lambda\eta^2-\tilde{\lambda}T^2)|\phi|^2+\frac{1}{4}\lambda\eta^4, \tag{8}$$

but ignoring the T^2 term, since initially $T=T_G<T_C=\sqrt{\lambda/\tilde{\lambda}}\eta$ and T decreases with time,

$$\partial^2\rho_B-(\partial_\mu\underline{\alpha}_B.\partial^\mu\underline{\alpha}_B)\rho_B-\frac{1}{2}\lambda\eta^2\rho_B = 0. \tag{9}$$

If we assume that in the initial configuration $\underline{\alpha}_B$ varies on some spatial scale $1/k$, this implies that

$$\partial_t^2\rho_B=m_I^2\rho_B\equiv(\frac{1}{2}\lambda\eta^2-k^2\underline{\alpha}_B^2)\rho_B \tag{10}$$

since we can take $\underline{\alpha}_B$ to be independent of time. We see that there is an exponential instability if $k^2\underline{\alpha}_B^2\le\frac{1}{2}\lambda\eta^2\equiv k_C^2$, corresponding to the modes that drive the transition at early times. Hence, for $k\alpha_B<k_C$,

$$\rho_B \simeq \mathcal{A}_B\exp\left(m_I(t-t_G)\right) \tag{11}$$

where $m_I^2\simeq k_C^2$.

Note that taking the initial configuration to fluctuate on a scale $1/k_C$, gives $\partial_i g \simeq k_C$, whilst, from above, $\dot{\rho_B}/\rho_B \simeq k_C$. This means that, for this period at least, the strength of the electric field will be of the same order of magnitude as the magnetic one.

Taking the assumption that the length-scale of fluctuations in $\underline{\alpha}_F$ is the same as that in ρ_F (we later demonstrate that this is self-consistent) we now show that $\rho_F/\rho_B < 1$.

Consider first-order perturbations to the equation of motion:

$$\partial^2 \rho_F - 2(\partial_\mu \underline{\alpha}_B).(\partial^\mu \underline{\alpha}_F)\rho_B - (\partial_\mu \underline{\alpha}_B).(\partial^\mu \underline{\alpha}_B)\rho_F - \frac{1}{2}\lambda\eta^2\rho_F + \lambda\rho_B^2\rho_F = -g\psi^2\rho_B \quad (12)$$

where we have introduced a thermal noise term by coupling the scalar, ϕ, to another, ψ, in thermal equilibrium.

Assuming that the wave number of fluctuations in α_F is less than k_C so that the second term in (12) does not dominate,

$$(\partial_t^2 - \nabla^2)\rho_F - m_I^2\rho_F \simeq -g\psi^2\rho_B \quad (13)$$

which can be solved by Green's function methods:

$$\rho_F(\underline{x}, t) = -g\int_0^t d\tau d^3y G_{ret}(t-\tau, \underline{x}-\underline{y})\psi^2(\tau, \underline{y})\rho_B(\tau, \underline{y}) \quad (14)$$

where

$$G_{ret}(x) = -\frac{1}{(2\pi)^4}\int d^4p \frac{\exp(-ipx)}{(p_0+i\epsilon)^2 - \underline{p}^2 + m_I^2}. \quad (15)$$

Now, for a rapid phase transition we can ignore the time-dependence of ψ^2. Substituting in the lower bound for ρ_B, we obtain

$$\rho_F(t, \underline{x}) \leq \frac{g\mathcal{A}_0}{2m_I^2}\psi^2 \exp(m_I t). \quad (16)$$

If we take ψ to be a self-interacting scalar field with self-coupling of order one, then in thermal equilibrium $\psi^2 \sim T^2$ and so

$$\rho_F/\rho_B \leq gT^2/m_I^2. \quad (17)$$

Provided that $\tilde{\lambda} > g$, initially $T = T_G < T_C = \sqrt{\lambda/\tilde{\lambda}}\eta \leq \sqrt{\lambda/g}\eta$ and so

$$\rho_F/\rho_B < 1. \tag{18}$$

As an aside, it is straightforward to show that

$$\partial_0\rho_F/\partial_0\rho_B \simeq \rho_F/\rho_B - g\left(\frac{T}{m_I}\right)^3\left(\frac{T}{m_{Pl}}\right) \tag{19}$$

and so $\partial_0\rho_F/\partial_0\rho_B < 1$. Since $\partial_0\rho/\rho \simeq \partial_0\rho_B/\rho_B + \partial_0\rho_F/\rho_B$ this suggests that, for its short lifetime, fluctuations have no major impact on the magnitude of the electric field.

For self-consistency, however, we still have to show that $\rho_F/\rho_B < 1$ implies $\partial_i\alpha_F < 1/k_C$. Our equation for $\underline{\alpha}_F$ is

$$\partial^2\underline{\alpha}_F - 2(\underline{k}_F.\underline{k}_C)\underline{\alpha}_F \quad = \quad 2(\underline{k}_C.\underline{k}_F)(\rho_F/\rho_B)\underline{\alpha}_B \tag{20}$$

Note that only $\underline{\alpha}_F$ and ρ_F are functions of $\underline{k}_F$. It is easy to Fourier transform and solve this using the Green's function method once more. Provided that $k_F \gg k_C$, we find that

$$\underline{\tilde{\alpha}}_F(\underline{k}_F, \tau) \sim \frac{\underline{k_C.k_F}}{k_F^2}\frac{\tilde{\rho_F}}{\rho_B}\underline{\alpha}_B \tag{21}$$

where a tilde denotes a Fourier transform, and so short wavelengths are suppressed compared to long wavelength inhomogeneities. Thus, thermal fluctuations do not destroy the generated magnetic field.

Conclusions

We have seen then that magnetic fields produced by gradients in the Higgs field between uncorrelated domains following a second order phase transition are stable to thermal fluctuations, despite not being of a topological nature. Whether such fields can then survive the ensuing microhydrodynamical processes to achieve the required magnitude today is a question that still needs to be answered, and one which depends heavily upon correct

estimates of the initial strength of the field. As shown, there is still some doubt about which of the current estimates, if any, is the correct one.

The case of a first order phase transition is even worse understood. We have seen how we might expect the field produced to be generally smaller in magnitude than one produced in an equivalent second order transition. For a *weakly* first order transition we might expect the formation and stability to differ little from the second order case. A detailed analysis is yet to be done however.

Another feature that we have observed is the transitive electric field produced as the scalar field drops into the minimum of the potential. Although, at least for part of its brief life, it is of the same strength as the magnetic field, whether it survives for long enough to imprint an induced magnetic field on the background is still an open question. However, the magnitude of the induced field is unlikely to be larger than the magnetic field considered here, and the same stability analysis applies.

More generally, although we take here the specific case of the electro-magnetic field, produced by the electro-weak phase transition, it should be noted that using the 't Hooft definition, magnetic fields are possible remnants of any phase transition in which a non-Abelian symmetry is broken, the field being associated with the unbroken Abelian[4] generators. Hence magnetic fields may exist associated with hypercharge, for example.

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